Classical tests of general relativity in brane world models

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The classical tests of general relativity (perihelion precession, deflection of light, and the radar echo delay) are considered for several spherically symmetric static vacuum solutions in brane world models. Generally, the spherically symmetric vacuum solutions of the brane gravitational field equations, have properties quite distinct as compared to the standard black hole solutions of general relativity. As a first step a general formalism that facilitates the analysis of general relativistic Solar System tests for any given spherically symmetric metric is developed. It is shown that the existing observational Solar System data on the perihelion shift of Mercury, on the light bending around the Sun (obtained using long-baseline radio interferometry), and ranging to Mars using the Viking lander, constrain the numerical values of the parameters of the specific models.

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I. INTRODUCTION

The idea, proposed in [1], that our four-dimensional Universe might be a three-brane, embedded in a five-dimensional space-time (the bulk), has attracted considerable interest in the past few years. According to the brane world scenario, the physical fields (electromagnetic, Yang-Mills, etc) in our four-dimensional Universe are confined to the three-brane. These fields are assumed to arise as fluctuations of branes in string theories. Only gravity can freely propagate in both the brane and bulk space-times, with the gravitational self-couplings not significantly modified. This model originated from the study of a single 3-brane embedded in five dimensions, with the 5D metric given by $ds^2 = e^{-f(y)}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2$, which due to the appearance of the warp factor $f(y)$, could produce a large hierarchy between the scale of particle physics and gravity. Even if the fifth dimension is uncompactified, standard 4D gravity is reproduced on the brane. Hence this model allows the presence of large, or even infinite non-compact extra dimensions. Our brane is identified to a domain wall in a 5-dimensional anti-de Sitter space-time. Due to the correction terms coming from the extra dimensions, significant deviations from the Einstein theory occur in brane world models at very high energies [2, 3]. Gravity is largely modified at the electro-weak scale 1 TeV. The cosmological and astrophysical implications of the brane world theories have been extensively investigated in the literature [4]. Gravitational collapse can also produce high energies, with the five dimensional effects playing an important role in the formation of black holes [5].

In standard general relativity the unique exterior space-time of a spherically symmetric object is described by the Schwarzschild metric. In the five dimensional brane world models, the high energy corrections to the energy density, together with the Weyl stresses from bulk gravitons, imply that on the brane the exterior metric of a static star is no longer the Schwarzschild metric [6]. The presence of the Weyl stresses also means that the matching conditions do not have a unique solution on the brane; the knowledge of the five-dimensional Weyl tensor is needed as a minimum condition for uniqueness. Static, spherically symmetric exterior vacuum solutions of the brane world models were first proposed in [6] and in [7]. The first of these solutions, obtained in [6], has the mathematical form of the Reissner-Nordstrom solution of standard general relativity, in which a tidal Weyl param-

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eter plays the role of the electric charge of the general relativistic solution. The solution was obtained by imposing the null energy condition on the 3-brane for a bulk having non zero Weyl curvature, and it can be matched to the interior solution corresponding to a constant density brane world star. A second exterior solution, which also matches a constant density interior, was derived in [7].

Several classes of spherically symmetric solutions of the static gravitational field equations in the vacuum on the brane have been obtained in [8–11]. As a possible physical application of these solutions the behavior of the angular velocity v_{tg} of the test particles in stable circular orbits has been considered [9–11]. The general form of the solution, together with two constants of integration, uniquely determines the rotational velocity of the particle. In the limit of large radial distances, and for a particular set of values of the integration constants the angular velocity tends to a constant value. This behavior is typical for massive particles (hydrogen clouds) outside galaxies, and is usually explained by postulating the existence of dark matter. The exact galactic metric, the dark radiation, the dark pressure and the lensing in the flat rotation curves region in the brane world scenario has been obtained in [11]. It is also interesting to note that the flat rotation curves can also be explained in $f(R)$ modified theories of gravity, without the need of dark matter [12].

Furthermore, two families of analytic solutions of the spherically symmetric vacuum brane world model equations (with $g_{tt} \neq -1/g_{rr}$), parameterized by the ADM mass and a PPN parameter β have been obtained in [13]. Non-singular black-hole solutions in the brane world model have been considered in [14], by relaxing the condition of the zero scalar curvature but retaining the null energy condition. The four-dimensional Gauss and Codazzi equations for an arbitrary static spherically symmetric star in a Randall–Sundrum type II brane world have been completely solved on the brane in [15]. The on-brane boundary can be used to determine the full 5-dimensional space-time geometry. The procedure can be generalized to solid objects such as planets. A method to extend into the bulk asymptotically flat static spherically symmetric brane world metrics has been proposed in [16]. The exact integration of the field equations along the fifth coordinate was done by using the multipole $(1/r)$ expansion. The results show that the shape of the horizon of the brane black hole solutions is very likely a flat "pancake" for astrophysical sources. The general solution to the trace of the 4-dimensional Einstein equations for static, spherically symmetric configurations has been used as a basis for finding a general class of black hole metrics, containing one arbitrary function $g_{tt} = A(r)$, which vanishes at some $r = r_h > 0$ (the horizon radius) in [17]. Under certain reasonable restrictions, black hole metrics are found, with or without matter. Depending on the boundary conditions the metrics can be asymptotically flat, or have any other prescribed asymptotic. For a review of the black hole properties and of the lensing in the brane world models see [18].

Now, to be viable models, the proposed models need to pass the astrophysical and cosmological observational tests. There are several possibilities of observationally testing the brane world models at an astrophysical/cosmological scale, such as using the time delay of gamma ray bursts [19] or by using the luminosity distance–redshift relation for supernovae at higher redshifts [20].

In addition to these possibilities, we also mention work on the gravitational lensing in brane world models [21], on the role of the brane charge in orbital models of high-frequency quasiperiodic oscillations observed in neutron star binary systems [22], on the complete set of analytical solutions of the geodesic equation of massive test particles in higher dimensional spacetimes which can be applied to brane world models [23], and on brane world corrections to the charged rotating black holes, to the perturbations in the electromagnetic potential and test particle motion around brane world black holes [24]. The classical tests of general relativity, namely, light deflection, time delay and perihelion shift, have been analyzed, for gravitational theories with large non-compactified extra-dimensions, in the framework of the five-dimensional extension of the Kaluza-Klein theory, using an analogue of the four-dimensional Schwarzschild metric in [25]. Solar system data also imposes some strong constraints on Kaluza-Klein type theories. The existence of extra-dimensions and of the brane world models can also be tested via the gravitational radiation coming from primordial black holes, with masses of the order of the lunar mass, $M \sim 10^{-7} M_{\odot}$, which might have been produced when the temperature of the universe was around 1 TeV. If a significant fraction of the dark halo of our galaxy consists of these lunar mass black holes, a huge number of black hole binaries could exist. The detection of the gravitational waves from these binaries could confirm the existence of extra-dimensions [26].

It is the purpose of the present paper to consider the classical tests (perihelion precession, light bending and radar echo delay) of general relativity for static gravitational fields in the framework of brane world gravity. To do this we shall adopt for the geometry outside a compact, stellar type object (the Sun), specific static and spherically symmetric vacuum solutions in the context of brane worlds. As a first step in our study, we consider the classical tests of general relativity in arbitrary spherically symmetric spacetimes, and develop a general formalism that can be used for any given metric. In particular, we consider the motion of a particle (planet), and analyze the perihelion precession, and in addition to this, by considering the motion of a photon, we study the bending of light by massive astrophysical objects and the radar echo delay, respectively. Existing data on light-bending around the Sun, using long-baseline radio interferometry, ranging to Mars using the Viking lander, and the perihelion precession of Mercury, can all give significant

and detectable Solar System constraints, associated with the brane world vacuum solutions. More precisely, the study of the classical general relativistic tests, constrain the parameters of the various solutions analyzed.

This paper is organized in the following manner. In Section II, we outline, for self-completeness and self-consistency, the field equations in brane world models. In Sec. III, we consider the classical Solar System tests in general relativity, namely, the perihelion shift, the light deflection and the radar echo delay, for arbitrary spherically symmetric spacetimes. In Sec. IV, we analyze the classical Solar System tests for the case of various brane world vacuum solutions. We conclude and discuss our results in Sec. V.

II. GRAVITATIONAL FIELD EQUATIONS ON THE BRANE

We start by considering a 5D spacetime (the bulk), with a single 4D brane, on which matter is confined. The 4D brane world $({}^{(4)}M, g_{\mu\nu})$ is located at a hypersurface $\left(B\left(X^A\right)=0\right)$ in the 5D bulk spacetime $({}^{(5)}M, g_{AB})$, where the coordinates are described by $X^A, A=0,1,\ldots,4$. The induced 4D coordinates on the brane are $x^\mu, \mu=0,1,2,3$.

The action of the system is given by [2]

$$S=S_{bulk}+S_{brane}, \tag{1}$$

where

$$S_{bulk}=\int_{{}^{(5)}M}\sqrt{-{}^{(5)}g}\left[\frac{1}{2k_5^2}{}^{(5)}R+{}^{(5)}L_m+\Lambda_5\right]d^5X, \tag{2}$$

and

$$S_{brane}=\int_{{}^{(4)}M}\sqrt{-{}^{(5)}g}\left[\frac{1}{k_5^2}K^{\pm}+L_{brane}\left(g_{\alpha\beta},\psi\right)+\lambda_b\right]d^4x, \tag{3}$$

where $k_5^2=8\pi G_5$ is the 5D gravitational constant; ${}^{(5)}R$ and ${}^{(5)}L_m$ are the 5D scalar curvature and the matter Lagrangian in the bulk, $L_{brane}\left(g_{\alpha\beta},\psi\right)$ is the 4D Lagrangian, which is given by a generic functional of the brane metric $g_{\alpha\beta}$ and of the matter fields ψ; $K^{\pm}$ is the trace of the extrinsic curvature on either side of the brane; and Λ_5 and λ_b (the constant brane tension) are the negative vacuum energy densities in the bulk and on the brane, respectively.

The energy-momentum tensor of bulk matter fields is defined as

$${}^{(5)}\tilde{T}_{IJ}\equiv-2\frac{\delta^{(5)}L_m}{\delta^{(5)}g^{IJ}}+{}^{(5)}g_{IJ}{}^{(5)}L_m\,, \tag{4}$$

while $T_{\mu\nu}$ is the energy-momentum tensor localized on the brane and is given by

$$T_{\mu\nu}\equiv-2\frac{\delta L_{brane}}{\delta g^{\mu\nu}}+g_{\mu\nu}\;L_{brane}. \tag{5}$$

Thus, the Einstein field equations in the bulk are given by [2]

$${}^{(5)}G_{IJ}=k_5^{2(5)}T_{IJ}\,, \tag{6}$$

with

$${}^{(5)}T_{IJ}=-\Lambda_5{}^{(5)}g_{IJ}+\delta(B)\left[-\lambda_b{}^{(5)}g_{IJ}+T_{IJ}\right]. \tag{7}$$

The delta function $\delta\left(B\right)$ denotes the localization of brane contribution. In the 5D spacetime a brane is a fixed point of the Z_2 symmetry. The basic equations on the brane are obtained by projections onto the brane world. The induced 4D metric is $g_{IJ}={}^{(5)}g_{IJ}-n_In_J$, where n_I is the space-like unit vector field normal to the brane hypersurface ${}^{(4)}M$. In the following we assume ${}^{(5)}L_m=0$.

Assuming a metric of the form $ds^2=(n_In_J+g_{IJ})dx^Idx^J$, with $n_Idx^I=d\chi$ the unit normal to the $\chi=$ constant hypersurfaces and g_{IJ} the induced metric on $\chi=$ constant hypersurfaces, the effective 4D gravitational equation on the brane (the Gauss equation), takes the form [2]:

$$G_{\mu\nu}=-\Lambda g_{\mu\nu}+k_4^2T_{\mu\nu}+k_5^4S_{\mu\nu}-E_{\mu\nu}, \tag{8}$$

where $S_{\mu\nu}$ is the local quadratic energy-momentum correction

$$S_{\mu\nu}=\frac{1}{12}TT_{\mu\nu}-\frac{1}{4}T_\mu{}^\alpha T_{\nu\alpha}+\frac{1}{24}g_{\mu\nu}\left(3T^{\alpha\beta}T_{\alpha\beta}-T^2\right), \tag{9}$$

and $E_{\mu\nu}$ is the non-local effect from the free bulk gravitational field, the transmitted projection of the bulk Weyl tensor C_{IAJB}, $E_{IJ}=C_{IAJB}n^An^B$, with the property $E_{IJ}\to E_{\mu\nu}\delta_I^\mu\delta_J^\nu$ as $\chi\to 0$. We have also denoted $k_4^2=8\pi G$, with G the usual 4D gravitational constant. The 4D cosmological constant, Λ, and the 4D coupling constant, k_4, are related by $\Lambda=k_5^2(\Lambda_5+k_5^2\lambda_b^2/6)/2$ and $k_4^2=k_5^4\lambda_b/6$, respectively. In the limit $\lambda_b^{-1}\to 0$ we recover standard general relativity [2].

The Einstein equation in the bulk and the Codazzi equation also imply the conservation of the energy-momentum tensor of the matter on the brane, $D_\nu T_\mu{}^\nu=0$, where D_ν denotes the brane covariant derivative. Moreover, from the contracted Bianchi identities on the brane it follows that the projected Weyl tensor obeys the constraint $D_\nu E_\mu{}^\nu=k_5^4D_\nu S_\mu{}^\nu$.

The symmetry properties of $E_{\mu\nu}$ imply that in general we can decompose it irreducibly with respect to a chosen 4-velocity field u^μ as

$$E_{\mu\nu}=-k^4\left[U\left(u_\mu u_\nu+\frac{1}{3}h_{\mu\nu}\right)+P_{\mu\nu}+2Q_{(\mu}u_{\nu)}\right], \tag{10}$$

where $k=k_5/k_4$, $h_{\mu\nu}=g_{\mu\nu}+u_\mu u_\nu$ projects orthogonal to u^μ, the "dark radiation" term $U=-k^4E_{\mu\nu}u^\mu u^\nu$ is a scalar, $Q_\mu=k^4h_\mu^\alpha E_{\alpha\beta}$ is a spatial vector and $P_{\mu\nu}=-k^4\left[h_{(\mu}{}^\alpha h_{\nu)}{}^\beta-\frac{1}{3}h_{\mu\nu}h^{\alpha\beta}\right]E_{\alpha\beta}$ is a spatial, symmetric and trace-free tensor.

In the case of the vacuum state we have $\rho = p = 0$, $T_{\mu\nu} \equiv 0$, and consequently $S_{\mu\nu} = 0$. Therefore the field equation describing a static brane takes the form

$$R_{\mu\nu} = -E_{\mu\nu} + \Lambda g_{\mu\nu}, \tag{11}$$

with the trace R of the Ricci tensor $R_{\mu\nu}$ satisfying the condition $R = R_{\mu}^{\mu} = 4\Lambda$.

In the vacuum case $E_{\mu\nu}$ satisfies the constraint $D_{\nu}E_{\mu}{}^{\nu} = 0$. In an inertial frame at any point on the brane we have $u^{\mu} = \delta_0^{\mu}$ and $h_{\mu\nu} = \mathrm{diag}(0,1,1,1)$. In a static vacuum $Q_{\mu} = 0$ and the constraint for $E_{\mu\nu}$ takes the form [7]

$$\frac{1}{3}D_{\mu}U + \frac{4}{3}UA_{\mu} + D^{\nu}P_{\mu\nu} + A^{\nu}P_{\mu\nu} = 0, \tag{12}$$

where $A_{\mu} = u^{\nu}D_{\nu}u_{\mu}$ is the 4-acceleration. In the static spherically symmetric case we may choose $A_{\mu} = A(r)r_{\mu}$ and $P_{\mu\nu} = P(r)\left(r_{\mu}r_{\nu} - \frac{1}{3}h_{\mu\nu}\right)$, where $A(r)$ and $P(r)$ (the "dark pressure" although the name dark anisotropic stress might be more appropriate) are some scalar functions of the radial distance r, and r_{μ} is a unit radial vector [6].

In order to obtain results which are relevant to the Solar System dynamics, in the following we will restrict our study to the static and spherically symmetric metric given by

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2 d\Omega^2, \tag{13}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. Here θ and ϕ are the standard coordinates on the sphere, $t \in R$ and r ranges over an open interval $(r_{\min}, r_{\max})$ so that $-\infty \leq r_{\min} \leq r_{\max} \leq \infty$. We also assume that the functions $\nu(r)$ and $\lambda(r)$ are strictly positive and (at least piecewise) differentiable on the interval $(r_{\min}, r_{\max})$.

Then the gravitational field equations and the effective energy-momentum tensor conservation equation in the vacuum take the form [8–10]

$$-e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right) + \frac{1}{r^2} = \frac{48\pi G}{k_4^4\lambda_b}U + \Lambda, \tag{14}$$

$$e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = \frac{16\pi G}{k_4^4\lambda_b}(U + 2P) - \Lambda, \tag{15}$$

$$e^{-\lambda}\left(\nu'' + \frac{\nu'^2}{2} + \frac{\nu' - \lambda'}{r} - \frac{\nu'\lambda'}{2}\right) = \frac{32\pi G}{k_4^4\lambda_b}(U - P) - 2\Lambda, \tag{16}$$

$$\nu' = -\frac{U' + 2P'}{2U + P} - \frac{6P}{r(2U + P)}, \tag{17}$$

where we denoted $' = d/dr$.

Equation (14) can immediately be integrated to give

$$e^{-\lambda} = 1 - \frac{C_1}{r} - \frac{2GM_U(r)}{r} - \frac{\Lambda}{3}r^2, \tag{18}$$

where C_1 is an arbitrary constant of integration, and we denoted

$$M_U(r) = \frac{24\pi}{k_4^4\lambda_b}\int_0^r r^2 U(r)\, dr. \tag{19}$$

The function M_U is the gravitational mass corresponding to the dark radiation term (the dark mass). By substituting ν' given by Eq. (17) into Eq. (15) and with the use of Eq. (18) we obtain the following system of differential equations satisfied by the dark radiation term U, the dark pressure P and the dark mass M_U, describing the vacuum gravitational field, exterior to a massive body, in the brane world model:

$$\begin{aligned}\frac{dU}{dr} &= -\frac{(2U + P)\left[2GM + M_U + \alpha(U + 2P)r^3\right] - \frac{2}{3}\Lambda r^2}{r^2\left(1 - \frac{2GM}{r} - \frac{M_U}{r} - \frac{\Lambda}{3}r^2\right)} \\ &\quad -2\frac{dP}{dr} - \frac{6P}{r},\end{aligned} \tag{20}$$

and

$$\frac{dM_U}{dr} = 3\alpha r^2 U, \tag{21}$$

respectively.

The system of equations (20) and (21) can be transformed to an autonomous system of differential equations by means of the transformations

$$\begin{aligned} q &= \frac{2GM}{r} + \frac{M_U}{r} + \frac{\Lambda}{3}r^2, \quad \mu = 3\alpha r^2 U + 3r^2\Lambda, \\ p &= 3\alpha r^2 P - 3r^2\Lambda, \quad \theta = \ln r. \end{aligned} \tag{22}$$

We may call μ and p the "reduced" dark radiation and pressure, respectively.

With the use of the new variables given by Eqs. (22), Eqs. (20) and (21) become

$$\frac{dq}{d\theta} = \mu - q, \tag{23}$$

and

$$\frac{d\mu}{d\theta} = -\frac{(2\mu + p)\left[q + \frac{1}{3}(\mu + 2p)\right]}{1 - q} - 2\frac{dp}{d\theta} + 2\mu - 2p, \tag{24}$$

respectively

Equations (20) and (21), or equivalently, (23) and (24) may be called the structure equations of the vacuum on the brane. In order to close this system an "equation of state" relating the reduced dark radiation and the dark pressure terms is needed. Generally, this equation of state is given in the form $P = P(U)$. In the new variables the functional relation between dark radiation and dark pressure takes the form $p = 3\alpha\exp(2\theta)P\left[\mu/3\alpha\exp(2\theta)\right]$. We consider several specific solutions below in the context of the Solar System tests.

III. CLASSICAL TESTS OF GENERAL RELATIVITY IN ARBITRARY SPHERICALLY SYMMETRIC STATIC SPACE-TIMES

At the level of the Solar System there are three fundamental tests, which can provide important observational evidence for general relativity and its generalization, and for alternative theories of gravitation in flat space. These tests are the perihelion precession of Mercury, the deflection of light by the Sun and the radar echo delay observations. These tests have been used to successfully test the Schwarzschild solution of general relativity. In order to constrain the brane world models at the level of the Solar System, we first have to study these effects in spherically symmetric space-times with arbitrary metrics. In this Section, we develop a formalism that can be used for any given metric. This useful formalism was first used in the Solar System tests applied to a specific vacuum solution in Hořava-Lifshitz gravity [27].

A. The perihelion precession

The motion of a test particle in the gravitational field on the brane in the metric given by Eq. (13) can be derived from the variational principle

$$\delta \int \sqrt{e^{\nu}c^2\dot{t}^2 - e^{\lambda}\dot{r}^2 - r^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right)}ds = 0, \quad (25)$$

where the dot denotes d/ds. It may be verified that the orbit is planar, and hence we can set $\theta = \pi/2$ without any loss of generality. Therefore we will use ϕ as the angular coordinate. Since neither t nor ϕ appear explicitly in Eq. (25), their conjugate momenta yield constants of motion,

$$e^{\nu}c^2\dot{t} = E = \text{constant}, \qquad r^2\dot{\phi} = L = \text{constant}. \quad (26)$$

The constant E is related to energy conservation while the constant L is related to angular momentum conservation.

The line element Eq. (13) provides the following equation of motion for r

$$\dot{r}^2 + e^{-\lambda}r^2\dot{\phi}^2 = e^{-\lambda}\left(e^{\nu}c^2\dot{t}^2 - 1\right). \quad (27)$$

Substitution of $\dot{t}$ and $\dot{\phi}$ from Eqs. (26) yields the following relationship

$$\dot{r}^2 + e^{-\lambda}\frac{L^2}{r^2} = e^{-\lambda}\left(\frac{E^2}{c^2}e^{-\nu} - 1\right). \quad (28)$$

The change of variable $r = 1/u$ and the substitution $d/ds = Lu^2 d/d\phi$ transforms Eq. (28) into the form

$$\left(\frac{du}{d\phi}\right)^2 + e^{-\lambda}u^2 = \frac{1}{L^2}e^{-\lambda}\left(\frac{E^2}{c^2}e^{-\nu} - 1\right). \quad (29)$$

By formally representing $e^{-\lambda} = 1 - f(u)$, we obtain

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = f(u)u^2 + \frac{E^2}{c^2L^2}e^{-\nu-\lambda} - \frac{1}{L^2}e^{-\lambda} \equiv G(u). \quad (30)$$

By taking the derivative of the previous equation with respect to ϕ we find

$$\frac{d^2u}{d\phi^2} + u = F(u), \quad (31)$$

where

$$F(u) = \frac{1}{2}\frac{dG(u)}{du}. \quad (32)$$

A circular orbit $u = u_0$ is given by the root of the equation $u_0 = F(u_0)$. Any deviation $\delta = u - u_0$ from a circular orbit must satisfy the equation

$$\frac{d^2\delta}{d\phi^2} + \left[1 - \left(\frac{dF}{du}\right)_{u=u_0}\right]\delta = O\left(\delta^2\right), \quad (33)$$

which is obtained by substituting $u = u_0 + \delta$ in Eq. (31). Therefore, to first order in δ, the trajectory is given by

$$\delta = \delta_0 \cos\left(\sqrt{1 - \left(\frac{dF}{du}\right)_{u=u_0}}\phi + \beta\right), \quad (34)$$

where δ_0 and β are constants of integration. The angles of the perihelia of the orbit are the angles for which r is minimum and hence u or δ is maximum. Therefore, the variation of the orbital angle from one perihelion to the next is

$$\phi = \frac{2\pi}{\sqrt{1 - \left(\frac{dF}{du}\right)_{u=u_0}}} = \frac{2\pi}{1 - \sigma}. \quad (35)$$

The quantity σ defined by the above equation is called the perihelion advance. It represents the rate of advance of the perihelion. As the planet advances through ϕ radians in its orbit, its perihelion advances through $\sigma\phi$ radians. From Eq. (35), σ is given by

$$\sigma = 1 - \sqrt{1 - \left(\frac{dF}{du}\right)_{u=u_0}}, \quad (36)$$

or, for small $(dF/du)_{u=u_0}$, by

$$\sigma = \frac{1}{2}\left(\frac{dF}{du}\right)_{u=u_0}. \quad (37)$$

For a complete rotation we have $\phi \approx 2\pi(1+\sigma)$, and the advance of the perihelion is $\delta\phi = \phi - 2\pi \approx 2\pi\sigma$. In order to be able to perform effective calculations of the perihelion precession we need to know the expression of L as a function of the orbit parameters. Let's consider the motion of a planet on a Keplerian ellipse with semi-axis

a and b, where $b = a\sqrt{1-e^2}$, and e is the eccentricity of the orbit. The surface area of the ellipse is πab. Since the elementary oriented surface area of the ellipse is $d\vec{\sigma} = \left(\vec{r}\times d\vec{r}\right)/2$, the areolar velocity of the planet is $\left|d\vec{\sigma}/dt\right| = \left|\vec{r}\times d\vec{r}\right|/2 = r^2\left(d\phi/dt\right)/2 \approx \pi a^2\sqrt{1-e^2}/T$, where T is the period of the motion, which can be obtained from Kepler's third law as $T^2 = 4\pi^2 a^3/GM$. In the small velocity limit $ds \approx cdt$, and the conservation of the relativistic angular momentum gives $r^2 d\phi/dt = cL$. Therefore we obtain $L = 2\pi a^2\sqrt{1-e^2}/cT$ and $1/L^2 = c^2/GMa\left(1-e^2\right)$.

As a first application of the present formalism we consider the precession of the perihelion of a planet in the Schwarzschild geometry with $e^{\nu} = e^{-\lambda} = 1-2GM/c^2 r = 1-\left(2GM/c^2\right)u$. Hence $f(u) = \left(2GM/c^2\right)u$. Since for this geometry $\nu+\lambda = 0$, we obtain first

$$G(u) = \frac{2GM}{c^2}u^3 + \frac{1}{L^2}\left(\frac{E^2}{c^2}-1\right) + \frac{2GM}{c^2L^2}u, \tag{38}$$

and then

$$F(u) = 3\frac{GM}{c^2}u^2 + \frac{GM}{c^2L^2}. \tag{39}$$

The radius of the circular orbit u_0 is obtained as the solution of the equation

$$u_0 = 3\frac{GM}{c^2}u_0^2 + \frac{GM}{c^2L^2}, \tag{40}$$

with the only physical solution given by

$$u_0 = \frac{1\pm\sqrt{1-12G^2M^2/c^4L^2}}{6GM/c^2} \approx \frac{GM}{c^2L^2}. \tag{41}$$

Therefore

$$\delta\phi = \pi\left(\frac{dF}{du}\right)_{u=u_0} = \frac{6\pi GM}{c^2 a\left(1-e^2\right)}, \tag{42}$$

which is the standard general relativistic result.

B. The deflection of light

In the absence of external forces a photon follows a null geodesic, $ds^2 = 0$. The affine parameter along the photon's path can be taken as an arbitrary quantity, and we denote again by a dot the derivatives with respect to the arbitrary affine parameter. There are two constants of motion, the energy E and the angular momentum L, given by Eqs. (26).

The equation of motion of the photon is

$$\dot{r}^2 + e^{-\lambda}r^2\dot{\phi}^2 = e^{\nu-\lambda}c^2\dot{t}^2, \tag{43}$$

which, with the use of the constants of motion can be written as

$$\dot{r}^2 + e^{-\lambda}\frac{L^2}{r^2} = \frac{E^2}{c^2}e^{-\nu-\lambda}. \tag{44}$$

The change of variable $r = 1/u$ and the use of the conservation equations to eliminate the derivative with respect to the affine parameter leads to

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = f(u)u^2 + \frac{1}{c^2}\frac{E^2}{L^2}e^{-\nu-\lambda} \equiv P(u). \tag{45}$$

By taking the derivative of the previous equation with respect to ϕ we find

$$\frac{d^2u}{d\phi^2} + u = Q(u), \tag{46}$$

where

$$Q(u) = \frac{1}{2}\frac{dP(u)}{du}. \tag{47}$$

In the lowest approximation, in which the term of the right hand side of the equation (46) is neglected, the solution is a straight line,

$$u = \frac{\cos\phi}{R}, \tag{48}$$

where R is the distance of the closest approach to the mass. In the next approximation Eq. (48) is used on the right-hand side of Eq. (46), to give a second order linear inhomogeneous equation of the form

$$\frac{d^2u}{d\phi^2} + u = Q\left(\frac{\cos\phi}{R}\right). \tag{49}$$

with a general solution given by $u = u\left(\phi\right)$. The light ray comes in from infinity at the asymptotic angle $\phi = -\left(\pi/2+\varepsilon\right)$ and goes out to infinity at an asymptotic angle $\phi = \pi/2+\varepsilon$. The angle ε is obtained as a solution of the equation $u\left(\pi/2+\varepsilon\right) = 0$, and the total deflection angle of the light ray is $\delta = 2\varepsilon$.

In the case of the Schwarzschild metric we have $\nu+\lambda = 0$ and $f(u) = \left(2GM/c^2\right)u$, which gives $P(u) = \left(2GM/c^2\right)u^3$ and $Q(u) = \left(3GM/c^2\right)u^2$, respectively. In the lowest approximation order from Eqs. (48) and (49) we obtain the second order linear equation

$$\frac{d^2u}{d\phi^2} + u = \frac{3GM}{c^2R^2}\cos^2\phi = \frac{3GM}{2c^2R^2}\left(1+\cos 2\phi\right), \tag{50}$$

with the general solution given by

$$u = \frac{\cos\phi}{R} + \frac{3GM}{2c^2R^2}\left(1-\frac{1}{3}\cos 2\phi\right). \tag{51}$$

By substituting $\phi = \pi/2+\varepsilon$, $u = 0$ into Eq. (51) we obtain

$$\varepsilon = \frac{2GM}{c^2R}, \tag{52}$$

where we have used the relations $\cos\left(\pi/2+\varepsilon\right) = -\sin\varepsilon$, $\cos\left(\pi+2\varepsilon\right) = -\cos 2\varepsilon$, $\sin\varepsilon \approx \varepsilon$ and $\cos 2\varepsilon \approx 1$. The total deflection angle in Schwarzschild geometry is $\delta = 2\varepsilon = 4GM/c^2R$.

C. Radar echo delay

A third Solar System test of general relativity is the radar echo delay [28]. The idea of this test is to measure the time required for radar signals to travel to an inner planet or satellite in two circumstances: a) when the signal passes very near the Sun and b) when the ray does not go near the Sun. The time of travel of light between two planets, situated far away from the Sun, is given by

$$T_0 = \int_{-l_1}^{l_2} \frac{dy}{c}, \tag{53}$$

where l_1 and l_2 are the distances of the planets to the Sun. If the light travels close to the Sun, the time travel is

$$T = \int_{-l_1}^{l_2} \frac{dy}{v} = \frac{1}{c} \int_{-l_1}^{l_2} e^{[\lambda(r) - \nu(r)]/2} dy, \tag{54}$$

where $v = ce^{(\nu-\lambda)/2}$ is the speed of light in the presence of the gravitational field. The time difference is

$$\delta T = T - T_0 = \frac{1}{c} \int_{-l_1}^{l_2} \left\{ e^{[\lambda(r) - \nu(r)]/2} - 1 \right\} dy. \tag{55}$$

Since $r = \sqrt{y^2 + R^2}$, we have

$$\delta T = \frac{1}{c} \int_{-l_1}^{l_2} \left\{ e^{\left[\lambda\left(\sqrt{y^2+R^2}\right) - \nu\left(\sqrt{y^2+R^2}\right)\right]/2} - 1 \right\} dy. \tag{56}$$

In the case of the Schwarzschild metric $\lambda = -\nu$, $\exp(\lambda/2 - \nu/2) = \exp(-\nu) = \left(1 - 2GM/c^2 r\right)^{-1} \approx 1 + 2GM/c^2 r$, and therefore

$$\delta T = \frac{2GM}{c^3} \int_{-l_1}^{l_2} \frac{dy}{\sqrt{y^2 + R^2}} = \frac{2GM}{c^3} \ln \frac{\sqrt{R^2 + l_2^2} + l_2}{\sqrt{R^2 + l_1^2} - l_1}. \tag{57}$$

Since

$$\ln \frac{\sqrt{R^2 + l_2^2} + l_2}{\sqrt{R^2 + l_1^2} - l_1} \approx \ln \frac{4 l_1 l_2}{R^2}, \tag{58}$$

where we have used the conditions $R^2/l_1^2 \ll 1$ and $R^2/l_2^2 \ll 1$, respectively, the time delay is given by

$$\delta T = \frac{2GM}{c^3} \ln \frac{4 l_1 l_2}{R^2}. \tag{59}$$

IV. SOLAR SYSTEM TESTS OF SPECIFIC BRANE WORLD MODELS

The braneworld description of our universe entails a large extra dimension and a fundamental scale of gravity that might be lower by several orders of magnitude compared to the Planck scale [1]. It is known that the Einstein field equations in five dimensions admit more general spherically symmetric black holes on the brane than four-dimensional general relativity. Hence an interesting consequence of the brane world scenario is in the nature of spherically symmetric vacuum solutions to the brane gravitational field equations, which could represent black holes with properties quite distinct as compared to ordinary black holes in four dimensions. Such black holes are likely to have very diverse cosmological and astrophysical signatures. In the present Section we consider the Solar System test properties of several brane world black holes, which have been obtained by solving the vacuum gravitational field equations. There are many black hole type solutions on the brane, and in the following we analyze three particular examples. We will assume, for astrophysical constants, the values [28]:

$$\begin{aligned} M_\odot &= 1.989 \times 10^{30} \text{ kg}, \\ R_\odot &= 6.955 \times 10^{8} \text{ m} \\ c &= 2.998 \times 10^{8} \text{ m/s}, \\ G &= 6.67 \times 10^{-11} \text{ m}^3\text{kg}^{-1}\text{s}^{-2}, \\ a &= 57.91 \times 10^{9} \text{ m}, \\ e &= 0.205615 \end{aligned} \tag{60}$$

A. The DMPR brane world vacuum solution

The first brane solution we consider is a solution of the vacuum field equations, obtained by Dadhich, Maartens, Papadopoulos and Rezania in [6], which represent the simplest generalization of the Schwarzschild solution of general relativity. We call this type of brane black hole as the DMPR black hole. The Solar System tests for the DMPR solutions were extensively analyzed in [29], but we use the general and novel formalism developed above as a consistency check. For this solution the metric tensor components are given by

$$e^{\nu} = e^{-\lambda} = 1 - \frac{2m}{r} + \frac{Q}{r^2}, \tag{61}$$

where Q is the so-called tidal charge parameter. In the limit $Q \to 0$ we recover the usual general relativistic case. In terms of the general equations discussed in section II, this class of brane world spherical solution is characterized by a equation of state relating dark energy and pressure: $P = -2U$. The metric is asymptotically flat, with $\lim_{r\to\infty} \exp(\nu) = \lim_{r\to\infty} \exp(\lambda) = 1$. There are two horizons, given by

$$r_h^{\pm} = m \pm \sqrt{m^2 - Q}. \tag{62}$$

Both horizons lie inside the Schwarzschild horizon $r_s = 2m$, $0 \le r_h^- \le r_h^+ \le r_s$. In the brane world models there is also the possibility of a negative $Q < 0$, which leads to only one horizon r_{h+} lying outside the Schwarzschild horizon,

$$r_{h+} = m + \sqrt{m^2 + Q} > r_s. \tag{63}$$

In this case the horizon has a greater area than its general relativistic counterpart, so that bulk effects act to increase the entropy and decrease the temperature, and to strengthen the gravitational field outside the black hole.

1. Perihelion precession

For the DMPR black hole we have the following relevant functions:

$$f(u) = 2mu - Qu^2 \,, \tag{64}$$

$$G(u) = 2mu^3 - Qu^4 + \frac{E^2}{c^2L^2} - \frac{1}{L^2} + \frac{2mu}{L^2} - \frac{Qu^2}{L^2} \,, \tag{65}$$

and

$$F(u) = 3mu^2 - 2Qu^3 + \frac{m}{L^2} - \frac{Qu}{L^2} \,, \tag{66}$$

respectively.

u_0 can be obtained as solution of the algebraic equation

$$3mu_0^2 - u_0 + \frac{m}{L^2} = 2Qu_0^3 + \frac{Q}{L^2}u_0 \,, \tag{67}$$

which to first order may be approximated to $u_0 \approx GM/(c^2L^2)$, assuming that $Q/L^2 \ll 1$. We have used the following relationship $m = GM/c^2$. Thus, $\delta\phi$ takes the form

$$\delta\phi = \frac{6\pi GM}{c^2a(1-e^2)} - \frac{c^2\pi Q}{GMa(1-e^2)} \,, \tag{68}$$

which is consistent with the result outlined in [29]. The first term in Eq. (68) is the well-known general relativistic correction term for the perihelion precession, while the second term gives the correction due to the non-local effects arising from the Weyl curvature in the bulk.

The observed value of the perihelion precession of the planet Mercury is $\delta\varphi_{Obs} = 43.11 \pm 0.21$ arcsec per century [28]. The general relativistic formula for the precession, $\delta\varphi_{GR} = 6\pi GM/c^2a\left(1-e^2\right)$, gives, using numerical values given in (60), $\delta\varphi_{GR} = 42.94$ arcsec per century. Therefore, the difference $\Delta\varphi = \delta\varphi_{Obs} - \delta\varphi_{GR} = 0.17 \pm 0.21$ arcsec per century can be attributed to other effects. By assuming that $\Delta\varphi$ is entirely due to the modifications of the general relativistic Schwarzschild geometry as a result of the five dimensional bulk effects, the observational results impose the following general constraint on the bulk tidal parameter Q:

$$|Q| \leq \frac{GM_{\odot}a\left(1-e^2\right)}{\pi c^2}\Delta\varphi \,. \tag{69}$$

With the use of the observational data for Mercury, Eq. (69) gives $|Q| \leq (5.17 \pm 6.39) \times 10^4$ m^2, or in the natural system of units, with $c = \hbar = G = 1$, $|Q| \leq (1.32 \pm 1.63) \times 10^{30}$ MeV^{-2}.

2. Deflection of light

We have the following functions

$$P(u) = 2mu^3 - Qu^4 + \frac{E^2}{c^2L^2}, \tag{70}$$

and

$$Q(u) = 3mu^2 - 2Qu^3 \,. \tag{71}$$

Solving the differential equation

$$\frac{d^2u}{d\phi^2} + u = \frac{3m}{R^2}\cos^2\phi - \frac{2Q}{R^3}\cos^3\phi \,, \tag{72}$$

provides the following exact solution

$$\begin{aligned} u(\phi) &= \frac{\cos\phi}{R} + \frac{3m}{2R^2}\left(1 - \frac{1}{3}\cos 2\phi\right) \\ &\quad - \frac{Q}{4R^3}\left(\frac{9}{4}\cos\phi - \frac{1}{4}\cos 3\phi + 3\phi\sin\phi\right) . \end{aligned} \tag{73}$$

By substituting $\phi = \pi/2 + \varepsilon$, $u = 0$ into Eq. (73), yields the following total deflection angle

$$\delta\phi = 2\varepsilon = \frac{4GM}{c^2R} - \frac{3\pi Q}{4R^2} = \frac{4GM}{c^2R}\left(1 - \frac{3\pi Qc^2}{16GMR}\right) \,, \tag{74}$$

which is in agreement with the results obtained in [30].

The best available data on light deflection by the Sun come from long baseline radio interferometry [31], which gives $\delta\phi_{LD} = \delta\phi_{LD}^{(GR)}\left(1+\Delta_{LD}\right)$, with $\Delta_{LD} \leq 0.0002 \pm 0.0008$, where $\delta\phi_{LD}^{(GR)} = 1.7510$ arcsec. Therefore light deflection constrains the tidal parameter Q as $|Q| \leq 16GMR\Delta_{LD}/3\pi c^2$. By taking for R the value of the radius of the Sun in (60), the light deflection gives the constraint $|Q| \leq (6.97 \pm 27.88) \times 10^8$ m^2, or, in natural units, $|Q| \leq (1.78 \pm 7.11) \times 10^{33}$ MeV^{-2}.

3. Radar echo delay

The delay can be evaluated from the integral in Eq. (56), with $\lambda = -\nu$, $\exp\left(\lambda/2 - \nu/2\right) = \exp\left(\lambda\right)$, so that

$$\exp\left(\lambda\right) = \left(1 - \frac{2GM}{c^2r} + \frac{Q}{r^2}\right)^{-1} \approx \left(1 + \frac{2GM}{c^2r} - \frac{Q}{r^2}\right) \,, \tag{75}$$

so that the time delay, given by Eq. (56), is readily integrated to yield

$$\begin{aligned} \delta T &= \frac{2GM}{c^3}\ln\left(\frac{\sqrt{R^2+l_2^2}+l_2}{\sqrt{R^2+l_1^2}-l_1}\right) \\ &\quad - \frac{Q}{cR}\left[\arctan\left(\frac{l_2}{R}\right) + \arctan\left(\frac{l_1}{R}\right)\right] . \end{aligned} \tag{76}$$

Using the approximations $R^2/l_1^2 \ll 1$ and $R^2/l_2^2 \ll 1$, the above expression reduces to

$$\delta T \approx \frac{2GM}{c^3} \ln\left(\frac{4l_1 l_2}{R^2}\right) - \frac{\pi Q}{cR}\,. \tag{77}$$

Recently the measurements of the frequency shift of radio photons to and from the Cassini spacecraft as they passed near the Sun have greatly improved the observational constraints on the radio echo delay. For the time delay of the signals emitted on Earth, and which graze the Sun, one obtains $\Delta t_{RD} = \Delta t_{RD}^{(GR)}(1+\Delta_{RD})$, with $\Delta_{RD} \simeq (1.1 \pm 1.2) \times 10^{-5}$ [32]. Therefore radar echo delay constrains the tidal charge of the DMPR brane world black hole as $|Q| \leq 2GMR\Delta_{RD} \ln\left(4l_1 l_2/R^2\right)/\pi c^2 \approx (1.83 \pm 1.99) \times 10^8$ m^2. In natural units we have $|Q| \leq (4.66 \pm 5.08) \times 10^{33}$ MeV^{-2}.

B. The CFM solution

Two families of analytic solutions in the brane world model, parameterized by the ADM mass and the PPN parameters β and γ, and which reduce to the Schwarzschild black hole for $\beta = 1$, have been found by Casadio, Fabbri and Mazzacurati in [13]. We denote the corresponding brane black hole solutions as the CFM black holes.

The first class of solutions is given by

$$e^{\nu} = 1 - \frac{2m}{r}, \tag{78}$$

and

$$e^{\lambda} = \frac{1 - \frac{3m}{2r}}{\left(1 - \frac{2m}{r}\right)\left[1 - \frac{3m}{2r}\left(1 + \frac{4}{9}\eta\right)\right]}, \tag{79}$$

respectively, where $\eta = \gamma - 1 = 2(\beta - 1)$. This solution is characterized by an equation of state of the form

$$\frac{P}{U} = \frac{1 - \frac{3m}{4r}}{\frac{m}{3r}} \tag{80}$$

As in the Schwarzschild case the event horizon is located at $r = r_h = 2m$. The solution is asymptotically flat, that is $\lim_{r\to\infty} e^{\nu} = e^{\nu_\infty} = \lim_{r\to\infty} e^{\lambda} = e^{\lambda_\infty} = 1$. We consider this case in the analysis outlined below as the CFM1 solution.

The second class of solutions corresponding to brane world black holes obtained in [13] has the metric tensor components given by

$$e^{\nu} = \left[\frac{\eta + \sqrt{1 - \frac{2m}{r}(1+\eta)}}{1+\eta}\right]^2, \tag{81}$$

and

$$e^{\lambda} = \left[1 - \frac{2m}{r}(1+\eta)\right]^{-1}, \tag{82}$$

respectively. The equation of state describing this solution is $U = 0$ The metric is asymptotically flat. In the case $\eta > 0$, the only singularity in the metric is at $r = r_0 = 2m(1+\eta)$, where all the curvature invariants are regular. $r = r_0$ is a turning point for all physical curves. For $\eta < 0$ the metric is singular at $r = r_h = 2m/(1-\eta)$ and at r_0, with $r_h > r_0$, where r_h defines the event horizon. We consider this case in the analysis outlined below as the CFM2 solution.

1. Perihelion precession

The function $G(u)$ in Eq. (30) specializes, for the CFM1 metric and CFM2 metric in

$$\begin{aligned} G_1(u) &= \left[1 - \frac{(1-2mu)\left(1 - \frac{3}{2}\left(1+\frac{4}{9}\eta\right)mu\right)}{\left(1 - \frac{3}{2}mu\right)}\right]u^2 \\ &+ \frac{1}{L^2}\frac{(1-2mu)\left(1 - \frac{3}{2}\left(1+\frac{4}{9}\eta\right)mu\right)}{\left(1 - \frac{3}{2}mu\right)} \times \\ &\times \left[\frac{E^2}{c^2}\frac{1}{(1-2mu)} - 1\right], \end{aligned} \tag{83}$$

and

$$\begin{aligned} G_2(u) &= 2m(1+\eta)u^3 - \frac{1 - 2m(1+\eta)u}{L^2} \\ &+ \frac{E^2}{c^2L^2}(1+\eta)^2 \frac{1 - 2m(1+\eta)u}{\left(\eta + \sqrt{1 - 2m(1+\eta)u}\right)^2}, \end{aligned} \tag{84}$$

respectively.

The GR limit is obtained for $\eta = 0$, which gives the expected value of the PPN parameter $\gamma = 1$. The function $F(u)$ is quite complicated, but we are only interested in the first order corrections to the GR results. Since the terms m/L, η and $E^2/c^2 - 1$ are very small, we can expand $F(u)$ up to second order, and keep only the term of order $O(2)$ in the product of the infinitesimals. We get, for the CFM1 and CFM2 solutions, respectively:

$$F_1(u) \simeq \frac{m}{L^2} + \frac{3(1+\frac{\eta}{3})}{m}u^2\,, \tag{85}$$

$$F_2(u) \simeq \frac{m}{L^2} + \frac{3(1+\eta)}{m}u^2\,. \tag{86}$$

Both function can be written as

$$F_{1,2}(u) \equiv \frac{\tilde{m}_{1,2}}{\tilde{L}_{1,2}^2} + 3\tilde{m}_{1,2}u^2\,, \tag{87}$$

where the tilded parameters are defined as:

$$\tilde{m}_1 = (1+\frac{\eta}{3})m; \qquad \tilde{L}_1^2 = (1+\frac{\eta}{3})L^2\,,$$
$$\tilde{m}_2 = (1+\eta)m; \qquad \tilde{L}_1^2 = (1+\eta+1)L^2\,.$$

This allows us to use directly the result obtained in GR, substituting the values of the tilded parameters. The resulting perihelion precession is:

$$\delta\phi_1 \simeq \frac{6\pi GM}{c^2 a(1-e^2)} + \frac{2\pi GM}{c^2 a(1-e^2)}\eta\,, \tag{88}$$

$$\delta\phi_2 \simeq \frac{6\pi GM}{c^2 a(1-e^2)} + \frac{6\pi GM}{c^2 a(1-e^2)}\eta\,, \tag{89}$$

where we have used the value of the Schwarzschild radius of the sun $m = GM_\odot/c^2$, as mentioned above. As in the previous section, we consider the case of the planet Mercury and assume that the discrepancy between the predicted and the observed value is completely due to extradimension influence. We thus obtain a limit for the value of η and consequently for β. In the CFM2 case (which is the most constraining) we have the following restrictions:

$$|\eta| \le 0.004 \pm 0.005 \tag{90}$$

The η parameter for the CFM1 solution has a bound three times larger.

2. Deflection of light

The function $P(u)$ in Eq. (45) for the CFM1 and the CFM2 metric takes the following respective forms:

$$\begin{aligned} P_1(u) &= \left[1 - \frac{(1-2mu)\left(1-\frac{3}{2}\left(1+\frac{4}{9}\eta\right)mu\right)}{(1-\frac{3}{2}mu)}\right]u^2 \\ &+ \frac{E^2}{c^2L^2}\frac{\left[1-\frac{3}{2}\left(1+\frac{4}{9}\eta\right)mu\right]}{(1-\frac{3}{2}mu)}\,, \end{aligned} \tag{91}$$

$$\begin{aligned} P_2(u) &= 2m(1+\eta)u^3 + \frac{E^2}{c^2L^2}(1+\eta)^2 \times \\ &\times \frac{1-2m(1+\eta)u}{\left(\eta+\sqrt{1-2m(1+\eta)u}\right)^2}\,. \end{aligned} \tag{92}$$

The function $Q(u)$ of Eq. (46) can be expanded as:

$$\begin{aligned} Q_1(u) &\simeq -\frac{\eta}{3}\frac{E^2}{c^2}\frac{m}{L^2} - \eta\frac{E^2}{c^2}\frac{m^2}{L^2}u \\ &+3m\left[1+\frac{\eta}{3}-\frac{3}{4}\eta\frac{E^2}{c^2}\frac{m^2}{L^2}\right]u^2 + O^3(u)\,, \end{aligned} \tag{93}$$

and

$$\begin{aligned} Q_2(u) &\simeq -\eta\frac{E^2}{c^2}\frac{m}{L^2} - 3\eta\frac{E^2}{c^2}\frac{m^2}{L^2}u \\ &+3m\left[1+\eta-\frac{\eta+5}{2}\frac{E^2}{c^2}\frac{m^2}{L^2}\right]u^2 + O^3(u)\,, \end{aligned} \tag{94}$$

for the CFM1 and CFM2 solutions, respectively.

As stated before, at first order, we can take, for both cases, $Q(u) = 0$, so that the zeroth-order solution is

$$u^{(0)} = \frac{\cos(\phi)}{R}\,, \tag{95}$$

which means the light rays travel on straight lines. Assuming, again, that the infinitesimals η and m/L are comparable, the first order equation is the same obtained in standard GR, namely:

$$u^{(1)\prime\prime} + u^{(1)} = 3m\left(\frac{\cos\phi}{R}\right)^2 \tag{96}$$

which gives, of course, the standard GR result

$$u^{(1)} = \frac{m}{2R^2}\left[3-\cos(2\phi)\right]. \tag{97}$$

At second order Eq. (46) can be written as

$$\begin{aligned} u_1^{(2)\prime\prime} + u_1^{(2)} &= -\frac{\eta}{3}\frac{E^2}{c^2}\frac{m}{L^2} + 6\frac{m}{L}u^{(0)}u^{(1)} + \eta\frac{m}{L}u^{(0)2} \\ = -\frac{\eta}{3}\frac{E^2}{c^2}\frac{m}{L^2} &+ \frac{3m^2}{R^3}\cos(\phi)\left[2-cos(2\phi)\right] + m\eta\frac{\cos^2\phi}{R^2}, \end{aligned} \tag{98}$$

and

$$\begin{aligned} u_2^{(2)\prime\prime} + u_2^{(2)} &= -\eta\frac{E^2}{c^2}\frac{m}{L^2} + 6\frac{m}{L}u^{(0)}u^{(1)} + 3\eta\frac{m}{L}u^{(0)2} \\ = \eta\frac{E^2}{c^2}\frac{m}{L^2} &+ \frac{3m^2}{R^3}\cos(\phi)\left[2-cos(2\phi)\right] + 3m\eta\frac{\cos^2\phi}{R^2}, \end{aligned} \tag{99}$$

respectively for the CFM1 and CFM2 solutions. The solutions are provided by

$$\begin{aligned} u_1^{(2)} &= \frac{m^2}{16R^3}\Bigg[12\cos^3(\phi) - \frac{16}{3}\eta\frac{R}{m}\cos^2(\phi) \\ &+21\cos(\phi)+60\phi\sin(\phi)\Bigg] - \frac{\eta}{3}\frac{E^2}{c^2}\frac{m}{L^2} + \frac{2}{3}\eta\frac{m}{R^2}, \end{aligned} \tag{100}$$

and

$$\begin{aligned} u_2^{(2)} &= \frac{m^2}{16R^3}\Bigg[12\cos^3(\phi) - 16\eta\frac{R}{m}\cos^2(\phi) \\ &+21\cos(\phi)+60\phi\sin(\phi)\Bigg] - \eta\frac{E^2}{c^2}\frac{m}{L^2} + 2\eta\frac{m}{R^2}, \end{aligned} \tag{101}$$

respectively.

The deflection angle is obtained by the equation $u(\pi/2+\epsilon) = 0$ with $\delta\phi = 2\epsilon$. Assuming $L = RE/c$ and substituting again $m = GM_\odot/c^2$, we finally find:

$$\delta\phi_1 = 4\frac{GM_\odot}{c^2R}\left(1+\frac{15}{16}\pi\frac{GM_\odot}{c^2R}+\frac{\eta}{6}\right), \tag{102}$$

$$\delta\phi_2 = 4\frac{GM_\odot}{c^2R}\left(1+\frac{15}{16}\pi\frac{GM_\odot}{c^2R}+\frac{\eta}{2}\right). \tag{103}$$

Comparison with data from the long baseline radio interferometry experiments [28], allow us, as in the previous section, to place an upper limit for η (and thus for β) which is, for the CFM2 solution:

$$|\eta| \le 0.0004 \pm 0.0016 \tag{104}$$

whilst, as before, the η parameter for the CFM1 solution has a bound three times larger.

3. Radar echo delay

The delay can be evaluated from the integral in Eq. (56). The integrand, in the case of the CMF1 and CFM2 metrics, is

$$\exp\left(\tfrac{\lambda-\nu}{2}\right) \simeq 1 + \left(2\tfrac{\eta}{3}\right) mu + \left(4 - \tfrac{5}{6}\eta\right) m^2u^2 + O(u^3)\,,$$
$$\exp\left(\tfrac{\lambda-\nu}{2}\right) \simeq 1 + (2+\eta)mu + \left(4 + \tfrac{9}{2}\eta\right) m^2u^2 + O(u^3)\,,$$

respectively, so that the delay is given by

$$\delta T_1 \;=\; 2\frac{GM}{c^3}\log\frac{4l_1l_2}{R^2}\left(1+\frac{\eta}{6}+2\pi\frac{GM}{c^2R}\log^{-1}\frac{4l_1l_2}{R^2}\right) \tag{105}$$

and

$$\delta T_2 \;=\; 2\frac{GM}{c^3}\log\frac{4l_1l_2}{R^2}\left(1+\frac{\eta}{2}+2\pi\frac{GM}{c^2R}\log^{-1}\frac{4l_1l_2}{R^2}\right)\,, \tag{106}$$

These results are to be compared with data from the Cassini spacecraft [32] described in the previous section. We get a bound for η in the CFM2 model:

$$|\eta| \leq 0.000021 \pm 0.000024 \tag{107}$$

the η parameter for the CFM1 solution has again a bound three times larger.

So we can conclude that the CFM models add a correction to the relevant test quantities which are directly proportional to the PPN parameter that describes the solution themselves, plus additional terms quadratic in the ratio between the Schwarzschild radius and the typical dimension of the phenomenon.

C. The BMD solution

Several classes of brane world black hole solutions have been obtained by Bronnikov, Melnikov and Dehnen in [17] (for short the BMD black holes). A particular class of these models has the metric given by

$$e^{\nu} = \left(1-\frac{2Gm}{c^2r}\right)^{2/s}\,, \qquad e^{\lambda} = \left(1-\frac{2Gm}{c^2r}\right)^{-2}\,, \tag{108}$$

where $s \in \mathbb{N}$. The metric is asymptotically flat, and at $r = r_h = 2Gm/c^2$ these solutions have a double horizon.

It is also interesting to mention that the DMPR and CFM black holes, analyzed above, are part of the families of solutions classified by Bronnikov, Melnikov and Dehnen in [17].

1. Perihelion precession

For the BMD black hole solutions $f(u) = (4Gm/c^2)u - (2Gm/c^2)^2u^2$ and $F(u)$ results in an involved expression. The equation $u_0 = F(u_0)$ cannot be solved exactly. We therefore expand $F(u)$ in $1/c^2$ keeping only terms up to second order, which means up to terms $1/c^4$, and find

$$F(u) \;\simeq\; \frac{2Gm}{c^2L^2s}+\frac{8G^2m^2}{c^4L^2s^2}u-\frac{12G^2m^2}{c^4L^2s}u +\frac{6Gm}{c^2}u^2-\frac{8G^2m^2}{c^4}u^3. \tag{109}$$

Therefore $u_0 = F(u_0)$ now becomes a cubic equation for u_0. The physical solution in up to terms of the order $(Gm/c^2)^4$ is given by

$$u_0 = \frac{2Gm}{c^2L^2s}\left(1+\frac{8G^2m^2}{c^4L^2s^2}\right). \tag{110}$$

Hence, the perihelion precession is

$$\delta\phi = \frac{6\pi Gm}{a^2c^2(1-e^2)}\left(1+\frac{4+6s-3s^2}{3s^2}\right). \tag{111}$$

Comparison with observations yields

$$-0.004 < \frac{4+6s-3s^2}{3s^2} < 0.004\,, \tag{112}$$

which can be achieved by the following parameter choices

$$-0.528 < s < -0.527\,, \qquad \text{or} \qquad 2.519 < s < 2.536\,. \tag{113}$$

These inequalities place stringent constraints on the parameter s, and since $s \in \mathbb{N}$, the restrictions strongly violate Solar System constrains. Thus, we can safely conclude that the BDM solution is incompatible with Mercury's perihelion precession.

2. Deflection of light

As above, the functional form of $Q(u)$ is too complicated to solve the resulting second order differential equation. Therefore, we expand $Q(u)$ in $1/c^2$ and find

$$Q(u) \;\simeq\; \frac{2Gm}{c^2L^2s}-\frac{2Gm}{c^2L^2}+\frac{6Gm}{c^2}u^2-\frac{8G^2m^2}{c^4}u^3 +\left(\frac{8G^2m^2}{c^4L^2s^2}-\frac{12G^2m^2}{c^4L^2s}+\frac{4G^2m^2}{c^4L^2}\right)u\,, \tag{114}$$

where the u^2 corresponds to classical general relativity and all other terms are contribution from the brane. Despite this complicated form of $Q(u)$ one can solve analytically the differential equation

$$\frac{d^2u}{d\phi^2}+u = Q\left(\frac{\cos\phi}{R}\right), \tag{115}$$

which results in a rather lengthy expression. However, by evaluating the solution $u(\phi = \pi/2+\varepsilon) = 0$ and by assuming that ε is small, and performing a Taylor expansion, the deflection angle becomes

$$\delta = \frac{4GM}{c^2R}\left(2-(1/s-1)\frac{R^2}{L^2}\right), \tag{116}$$

where the factor up front corresponds to the case of general relativity. Long baseline radio interferometry constrains the factor in the bracket to be ≤ 1.0017 and therefore s would have to be a very small real number to yield compatibility with Solar System tests. This parameter choice would also contradict the perihelion precession results discussed above.

Thus, there is no need to further discuss the radar echo delay for this solution as it is already incompatible with Solar System constraints. Lastly, we would like to note that these conclusions also follow by considering the PPN parameters for the BMD solution, which are given by $\gamma = 2$ and $\beta = 2 + 4/s^2 - 2/s$. Clearly, $\gamma = 2$ violates Solar System constrain while the minimum value for β is given by $\beta_{\min} = 7/4$ which is attained when $s = 4$, again incompatible with Solar System tests.

V. DISCUSSIONS AND FINAL REMARKS

The study of the classical tests of general relativity provides a very powerful method for constraining the allowed parameter space of brane world solutions, and to provide a deeper insight into the physical nature and properties of the corresponding spacetime metrics. Therefore, this opens the possibility of testing braneworld gravity by using astronomical and astrophysical observations at the Solar System scale. In the present paper, we have developed a general formalism that facilitates the analysis of any given metric and provides the basic theoretical tools necessary for the in depth comparison of the predictions of the brane world models with the observational/experimental results. In this context, the classical tests of general relativity (perihelion precession, deflection of light, and the radar echo delay) were considered for specific static and spherically symmetric vacuum solutions in brane world models.

First, we analyzed a solution of the vacuum field equations, obtained by Dadhich, Maartens, Papadopoulos and Rezania (DMPR) [6], which represents the simplest generalization of the Schwarzschild solution of general relativity. The tightest limit we get on the parameter Q came from the perihelion precession of Mercury, and gives $|Q| \leq (1.32 \pm 1.63) \times 10^{30}$ MeV^{-2}, with other bounds about three orders of magnitude larger. These results represent a significant improvement of the results obtained in [29], where a method based on the first order approximation of the Hamilton-Jacobi result was used.

Second, we considered two families of analytic solutions, parameterized by the ADM mass and the PPN parameters β and γ, found by Casadio, Fabbri and Mazzacurati (CFM) [13]. These solutions can be physically seen as "dark pressure-dominated" solutions. Contrary to what happens in the DMPR case, here the most stringent bound came from the radar echo delay data of the Cassini experiment, bounding the relevant parameter η to be $|\eta| \leq 0.000021 \pm 0.000024$.

Thus, these first two models analyzed are not directly ruled out by Solar System tests. To get a value that is more "physically" meaningful, we can use the matching conditions between the interior and the exterior of a uniform density star in brane world, as derived in [7] to obtain the brane tension in terms of the relevant parameter of the different solutions. We get:

$$
\begin{aligned}
\lambda_b &\geq \frac{9GM_\odot^2}{4\pi|Q|R_\odot^2} \\
\lambda_b &\geq \frac{27M_\odot c^2\left(1-\frac{3GM_\odot}{c^2R_\odot}\right)}{8\pi R_\odot^3|\eta|} \\
\lambda_b &\geq \frac{9M_\odot c^2\left(\eta+\sqrt{1-\frac{2GM_\odot}{c^2R_\odot}(1+\eta)}\right)}{8\pi R_\odot^3|eta|(1+eta)} \qquad (117)
\end{aligned}
$$

for the DMPR, the CFM1 and the CFM2 solutions respectively. Interestingly enough, the most stringent bound comes from the DMPR solution (we are only interested in orders of magnitude here, so we will omit actual numerical values and errors): $\lambda_b \geq 10^7$ GeV^4, while the CFM solutions gives $\lambda_b \geq 10^4$ GeV^4. We do not have a definitive explanation for this, but we believe it is related to the fact that in the CFM solutions the dark radiation is basically negligible. Incidentally, the two different bounds obtained in the different CFM solutions lead to exactly the same bound on the brane tension, thus indicating that, although causally different, the two solutions behave in the same way for what concerns physical phenomena far away the central source.

In third place, we considered several classes of brane world black hole solutions obtained by Bronnikov, Melnikov and Dehnen [17] (BMD). The observational Solar System test place stringent constraints on the parameter space of the model, and these restrictions strongly violate Solar System constrains, so we can safely rule out these solutions as viable solutions.

Despite the fact of having a whole plethora of brane world vacuum solutions that pass the Solar System observational test a few remarks are in order, namely, one may basically consider two strategies of obtaining solutions on the brane. First, the bulk spacetime may be given, by solving the full 5-dimensional equations, and the geometry of the embedded brane is then deduced. Second, due to the complexity of the 5-dimensional equations, one may follow the strategy outlined in this paper, by considering the intrinsic geometry on the brane, which encompasses the imprint from the bulk, and consequently evolve the metric off the brane. In principle, the second procedure may provide a well-determined set of equations, with the brane setting the boundary data. However, determining the bulk geometry proves to be an extremely difficult endeavor.

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